Fundamental bounds on quantum measurements with a mixed apparatus

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We consider the apparatus in a quantum measurement process to be in a mixed state. We propose a simple upper bound on the probability of correctly distinguishing any number of mixed states. We use this to derive fundamental bounds on the efficiency of a measurement in terms of the temperature of the apparatus.

The quantum measurement process was first analysed from the fully quantum perspective by von Neumann [1]. In the description of a typical quantum measurement, the initial states of the system and the apparatus (both treated quantum mechanically) are taken to be pure states. The measurement process is a transformation of the type

$$\int \phi(x)|x\rangle_s dx \otimes \int \eta(y)|y\rangle_a dy$$
$$\rightarrow \int \int \phi(x)\eta(y+f(x))|x\rangle_s \otimes |y\rangle_a dx dy \quad (1)$$

where $\int \phi(x)|x\rangle_s dx$ and $\int \eta(y)|y\rangle_a dy$ are initial states of the system and the measuring apparatus respectively. Note that after the measurement, the state of the apparatus is correlated to the state of the system. This enables us to infer the state of the system by observing the state of the apparatus. In general, one is allowed to perform *any* positive operator valued measurement (POVM) on the apparatus state to infer the system state (not analysed by von Neumann). This becomes important in a realistic quantum measurement, in which the apparatus is a macroscopic system and likely to be in a mixed (generally thermal) state throughout the measurement. This will make measurements more difficult in general than the pure apparatus case as mixed states are generically harder to distinguish. One could, of course, attempt to estimate the efficiency of state inference on the basis of specific von Neumann projections on the apparatus state. This, however, is usually less efficient than a more general POVM. Strangely, to the best of our knowledge, all treatments of quantum measurement seem to neglect this fact and assume the initial apparatus state to be pure. In this letter, we analyse the quantum measurement process with a mixed apparatus. Our formalism allows us to estimate the probability of success in state determination with a mixed apparatus. We use this to put bounds on the probability os successful state inference in terms of the temperature of an apparatus in a thermal state.

We start with a simple example of a two level system being measured by a harmonic oscillator apparatus initially in a thermal state. The measurement interaction is described by

$$|0\rangle_s \otimes |n\rangle_a \rightarrow |0\rangle_s \otimes |n\rangle_a$$
$$|1\rangle_s \otimes |n\rangle_a \rightarrow |1\rangle_s \otimes |n+1\rangle_a, \quad (2)$$

where $|0\rangle_s, |1\rangle_s$ are two orthogonal states of the two level system that the apparatus is designed to detect and $|n\rangle_a$ denotes a Fock state of the apparatus. Note that we could have chosen any other measurement interaction, but that would lead to similar results. In particular, this measurement interaction works perfectly when the apparatus is initially in a pure Fock state. If the apparatus starts off in the initial thermal state $\rho(\beta)_a = \sum_{n=0}^{\infty} \frac{e^{-\beta n}}{Z}(|n\rangle\langle n|)_a$ (here Z is the partition function $\sum_{n=0}^{\infty} e^{-\beta n}$ and $\beta = \hbar\omega/k_B T$ where ω is the frequency of the oscillator, T its temperature and k_B is Boltzmann's constant), then the measurement leads to

$$(|0\rangle\langle 0|)_s \otimes \rho(\beta)_a \rightarrow (|0\rangle\langle 0|)_s \otimes \rho(\beta)_a$$
$$(|1\rangle\langle 1|)_s \otimes \rho(\beta)_a \rightarrow (|1\rangle\langle 1|)_s \otimes \sum_{n=1}^{\infty} \frac{e^{-\beta(n-1)}}{Z}(|n\rangle\langle n|)_a. \quad (3)$$

The maximum probability P_c of correctly distinguishing between any two mixed states ρ_0 and ρ_1 by any POVM is given by Helstrom's formula [2]

$$P_c(\rho_0, \rho_1) = \frac{1}{2} + \frac{1}{4}\text{Tr}|\rho_0 - \rho_1|. \quad (4)$$

In our case $\rho_0 = \rho(\beta)_a$ and $\rho_1 = \sum_{n=1}^{\infty} \frac{e^{-\beta(n-1)}}{Z}(|n\rangle\langle n|)_a$. We thus obtain the probability of correctly identifying the state of the apparatus (and hence the system, which is our main goal) to be

$$P_c = \frac{1}{2} + \frac{1}{4}\{\frac{1}{Z} + \frac{e^{\beta}-1}{Z}(Z-1)\}. \quad (5)$$
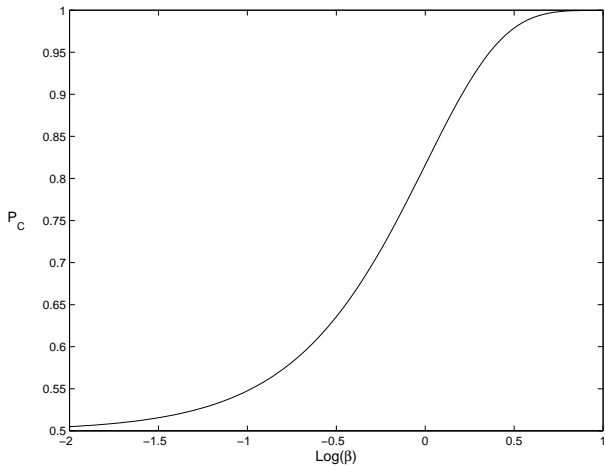
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FIG. 1. The figure shows the variation of the probability of correct state inference of a two level system being measured by a harmonic oscillator apparatus with $\log \beta$, where $\beta = \hbar\omega/k_B T$ and the logarithm is in base 10. Here P_c is obtained from Helstrom's formula.

We see from fig.1 that we obtain $P_c \to 1$ for $\beta \to \infty$ (low temperature limit) and $P_c \to 0.5$ for $\beta \to 0$ (high temperature limit). This means that at low temperatures we can correctly distinguish between $|0\rangle_s$ and $|1\rangle_s$ and this is because then the initial state of the apparatus is virtually pure. On the other hand for high temperatures, the initial state of the apparatus is virtually maximally mixed and does not change due to the measurement interaction (i.e $\rho_0 = \rho_1$). We see that for achieving P_c greater than 0.8, we require

$$\frac{\omega}{T} \geq \frac{k_B}{\hbar}. \tag{6}$$

This can be seen as a fundamental limit on obtaining one bit of information reliably.

The quantum system being measured, may, however, be a system with more than two orthogonal states. In that case we have to have a formula for the probability of correctly identifying any one of several mixed states $\rho_0, \rho_1, ..., \rho_N$. There is no existing general formula extending the Helstrom's formula to an arbitrary number of density matrices. Here we propose a generalisation which gives a bound on the probability of correct identification of one of $\rho_0, \rho_1, ..., \rho_N$ which appear with probabilities $p_0, p_1, ..., p_N$. This bound is

$$P_c(\rho_0, \rho_1, ..., \rho_N; p_0, p_1, ..., p_N) = e^{H - h(p_0, p_1, ..., p_N)}, \tag{7}$$

where

$$H = S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i), \tag{8}$$

is the Holevo bound [3], $S(\rho) = -\mathrm{Tr}\rho \ln \rho$ and where

$$h(p_0, p_1, ..., p_N) = -\sum_i p_i \ln p_i, \tag{9}$$

is the Shannon entropy of the probability distribution of the density matrices. The rationale behind this formula becomes clear when one considers a sequence of n preparations of the system state and measurements on the corresponding apparatus states. The probability of correctly inferring a certain sequence of n states $\{\rho_i\}$ is bounded above by the ratio of the number of correctly identified sequences and the total number of possible sequences. From the statistical interpretation of the quantum relative entropy [4], we get the numerator of this ratio to be e^{nH} and the denominator is $e^{nh(p_0, p_1, ..., p_N)}$ (this is equivalent to the law of large numbers). We, in fact, use the formula for $n = 1$ and hence the Eq.(7) for P_c.
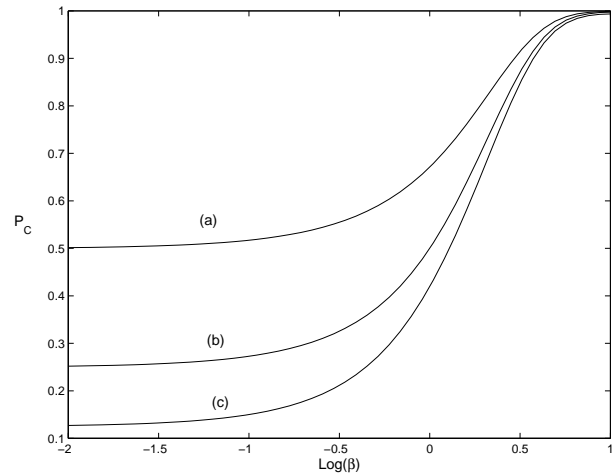

FIG. 2. The figure shows the variation of the upper bound on the probability of correct state inference of a $N + 1$ level system being measured by a harmonic oscillator apparatus with $\log \beta$, where $\beta = \hbar\omega/k_B T$ and the logarithm is in base 10. For plot(a) $N = 1$, for plot(b) $N = 3$, for plot(c) $N = 7$. Here P_c is obtained from our formula Eq.(7).

We again take a harmonic oscillator apparatus, but now a $N + 1$ level system with orthogonal states $|0\rangle_s, |1\rangle_s, ..., |N\rangle_s$ which interacts with the apparatus as

$$|i\rangle_s \otimes |n\rangle_a \to |i\rangle_s \otimes |n + i\rangle_a. \tag{10}$$

The initial state of the apparatus is again a thermal state. Corresponding to each system state $|i\rangle_s$, the apparatus will evolve to a different state ρ_a^i. These are given by

$$\rho_a^i = \sum_{n=i}^{\infty} \frac{e^{-\beta(n-i)}}{Z}(|n\rangle\langle n|)_a. \tag{11}$$

The entropy of all these states is the same and equal to

$$S(\rho_a^i) = \frac{\langle E \rangle}{k_B T} + \ln Z, \tag{12}$$

where $\langle E \rangle = (\hbar\omega/Z) \sum n e^{-\beta n}$ is the average energy of the apparatus.

The entropy of the total state $\rho = \sum_i \rho_a^i$ is given by

$$S(\rho) = -\sum_{i=0}^{N} \left(\sum_{j=0}^{i} \frac{e^{-\beta j}}{Z(N+1)}\right) \ln \left(\sum_{j=0}^{i} \frac{e^{-\beta j}}{Z(N+1)}\right)$$

$$- \sum_{i=N+1}^{\infty} \left(\sum_{j=i-N}^{i} \frac{e^{-\beta j}}{Z(N+1)}\right) \ln \left(\sum_{j=i-N}^{i} \frac{e^{-\beta j}}{Z(N+1)}\right) \quad (13)$$

The entropies $S(\rho)$ and $S(\rho_a^i)$ are used to compute P_c from Eq.(7). This is plotted in Fig. 2 for three different dimensionalities of the measured system ($N+1 = 2, 4, 8$). Note that all the three plots in Fig.2 have the same shape and, as expected, tend to $\frac{1}{N+1}$ for small β (high temperature limit) and to unity for large β (low temperature limit). Note also that in the latter case (when $\beta > 1$) the probability bound P_c is well approximated by the simple expression

$$P_c \sim \frac{e^{-\frac{\langle E \rangle}{k_B T}}}{Z}. \quad (14)$$

This is because, at low temperatures, $S(\rho) \sim h(p_0, p_1, ..., p_N)$, so that $P_c \sim e^{-S(\rho_a^0)}$. The above approximation (Eq.(14)) is already very good for $\beta = 5$ differing by about 2 percent from the exact value.

Our bound applies to the general setting of measuring states of a system by correlating them to pure nonorthogonal apparatus states. This happens when different orthogonal states of the system get correlated to different nonorthogonal states of the apparatus (for example, when different Fock states inside a cavity are inferred by different dispacements of a mirror in a coherent state [5]). In quantum optics, in particular, preparation of nonclassical states via conditional measurements on an apparatus is very popular [6]. Our formula for P_c will be an upper bound to the fidelity of such preparations. We hope that this kind of analysis also stimulates more research in the area of quantum computation with mixed states as explored in Ref. [7].

[1] J. von Neumann, *Mathematical foundations of quantum mechanics*, Princeton University Press, Princeton, 1955; H. Everett III, *The theory of the universal wave function* in *The many worlds interpretation of quantum mechanics* (B. S. DeWitt and N. Graham eds.), Princeton University Press, Princeton, 1973.

[2] C. W. Helstrom, *Quantum detection and estimation theory*, Academic Press, NY (1976).

[3] A. S. Kholevo, Probl. Peredachi Inf. **9**, 177 (1973).

[4] V. Vedral, M. B. Plenio, K. Jacobs and P. L. Knight, Phys. Rev. A **56** 4452 (1997).

[5] S. Bose, K. Jacobs and P. L. Knight, Phys. Rev. A **56**, 4175 (1997); *ibid* **59**, 3204 (1999) and references therein.

[6] K. Jacobs, P. L. Knight and V. Vedral, J. Mod. Opt. **44**, 2427 (1997) and references therein.

[7] S. Bose, L. Rallan and V. Vedral, *Communication capacity of quantum computation*, LANL e-print quant-ph/0003072.